

# Gauge Non-Invariant Higher-Spin Currents in $AdS_4$

P.A. Smirnov and M.A. Vasiliev

*I.E.Tamm Department of Theoretical Physics, Lebedev Physical Institute of RAS,  
Leninsky prospect 53, 119991, Moscow, Russia*

## Abstract

Conserved currents of any spin  $t > 0$  built from bosonic symmetric massless gauge fields of arbitrary integer spins in  $AdS_4$  are found. Analogously to the case of  $4d$  Minkowski space, currents considered in this paper are not gauge invariant but generate gauge invariant conserved charges.

# 1 Introduction

The gauge invariant conserved currents were studied in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9].

Generally, a conserved current carries three spins  $(t, s_1, s_2)$ , where  $t$  is a spin of the current, while  $s_1$  and  $s_2$  are spins of fields from which it is constructed. For example, stress tensor ( $t = 2$ ) exists for matter fields of arbitrary spins  $s_1 = s_2 = s$ . It is well known however that for  $s = t = 2$  the stress tensor is not gauge invariant corresponding to the so-called gravitational stress pseudo-tensor [13]. As shown by Deser and Waldron [5], for  $t = 2$  analogous phenomenon occurs for all massless fields of spins  $s > 2$ .  $t = 1$  currents have similar property. The currents with  $s < 1$  are gauge invariant (in this case, for a trivial reason since a spin-zero field has no gauge symmetry), while the spin-one current built from two massless spin-one fields is not.

The aim of this paper is to extend the results of [10] in Minkowski space, presenting the full list of gauge noninvariant currents with integer spins  $AdS_4$  such that the spin of current is smaller than the sum of spins of fields from which it is built. Being gauge non-invariant these currents give rise to the gauge invariant conserved charges. The convenient way to obtain such currents is via the variation of the cubic action of [11, 12].

## Conventions

In this paper  $AdS_4$  is considered. Greek indices  $\mu, \nu, \rho, \lambda, \sigma$  are base and range from 0 to 3. Other Greek indices are spinorial and take values 1, 2. The latter are raised and lowered by the  $sp(2)$  antisymmetric forms  $\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}, \varepsilon^{\dot{\alpha}\dot{\beta}}$

$$\varepsilon^{\alpha\beta}\varepsilon_{\alpha\gamma} = \delta_{\gamma}^{\beta}, \quad \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\gamma}} = \delta_{\dot{\gamma}}^{\dot{\beta}}, \quad (1)$$

$$A_{\alpha} = A^{\beta}\varepsilon_{\beta\alpha}, \quad A^{\alpha} = A_{\beta}\varepsilon^{\alpha\beta}, \quad A_{\dot{\alpha}} = A^{\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\alpha}}, \quad A^{\dot{\alpha}} = A_{\dot{\beta}}\varepsilon^{\dot{\alpha}\dot{\beta}}. \quad (2)$$

Complex conjugation  $\bar{A}$  relates dotted and undotted spinors. Brackets ( $[...]$ )  $\{...\}$  imply complete (anti)symmetrization, *i.e.*,

$$A_{[\alpha}B_{\beta]} = \frac{1}{2}(A_{\alpha}B_{\beta} - A_{\beta}B_{\alpha}), \quad A_{\{\alpha}B_{\beta\}} = \frac{1}{2}(A_{\alpha}B_{\beta} + A_{\beta}B_{\alpha}). \quad (3)$$

$A^{\alpha(m)}$  denotes a totally symmetric multispinor  $A^{\{\alpha_1 \dots \alpha_m\}}$ .

The wedge symbol  $\wedge$  is implicit.

## 2 Fields, equations, actions

In the case of  $4d$  space considered in this paper it is convenient to use the frame-like formalism in two-component spinor notation. In this formalism a bosonic spin- $s$  field of Fronsdal [14] is represented by multispinor one-forms [15]:

$$s \geq 2: \quad \varphi_{\mu_1 \dots \mu_s} \rightarrow \{\omega^{\alpha(m), \dot{\beta}(n)} \mid m+n=2(s-1)\}, \quad \omega^{\alpha(m), \dot{\beta}(n)} = dx^\mu \omega_\mu^{\alpha(m), \dot{\beta}(n)},$$

which are symmetric in all dotted and all undotted spinor indices and obey the reality condition [15]

$$\omega_{\alpha(m), \dot{\beta}(n)}^\dagger = \omega_{\beta(n), \dot{\alpha}(m)}. \quad (4)$$

The frame-like field is a particular connection at  $n = m = s - 1$

$$h_\mu^{\alpha(s-1), \dot{\beta}(s-1)} dx^\mu := \omega_\mu^{\alpha(s-1), \dot{\beta}(s-1)} dx^\mu. \quad (5)$$

By imposing appropriate constraints, the connections  $\omega^{\alpha(m), \dot{\beta}(n)}$  can be expressed via  $t = \frac{1}{2}|m-n|$  derivatives of the frame field [15].

Background gravity is described by the vierbein one-form  $\tilde{h}^\alpha_{\dot{\beta}}$  and one-form connections  $\tilde{\omega}^{\dot{\alpha}\dot{\beta}}, \tilde{\omega}^{\alpha\beta}$ . Lorentz covariant derivative  $\tilde{D}$  acts as usual

$$\tilde{D}A^{\alpha(m), \dot{\beta}(n)} = dA^{\alpha(m), \dot{\beta}(n)} + m\tilde{\omega}^\alpha_\gamma A^{\alpha(m-1)\gamma, \dot{\beta}(n)} + n\tilde{\omega}^{\dot{\beta}}_{\dot{\delta}} A^{\alpha(m), \dot{\beta}(n-1)\dot{\delta}} \quad (6)$$

for any multispinor  $A^{\alpha(m), \dot{\beta}(n)}$ . The torsion and curvature two-forms are

$$\tilde{R}^{\alpha, \dot{\beta}} = d\tilde{h}^{\alpha, \dot{\beta}} + \tilde{\omega}^\alpha_\gamma \tilde{h}^{\gamma, \dot{\beta}} + \tilde{\omega}^{\dot{\beta}}_{\dot{\delta}} \tilde{h}^{\alpha, \dot{\delta}}, \quad (7)$$

$$\tilde{R}^{\alpha\alpha} = d\tilde{\omega}^{\alpha\alpha} + \tilde{\omega}^\alpha_\gamma \tilde{\omega}^{\alpha\gamma} - \lambda^2 \tilde{\omega}^\alpha_{\dot{\delta}} \tilde{\omega}^{\alpha\dot{\delta}}, \quad (8)$$

$$\tilde{R}^{\dot{\beta}\dot{\beta}} = d\tilde{\omega}^{\dot{\beta}\dot{\beta}} + \tilde{\omega}^{\dot{\beta}}_{\dot{\gamma}} \tilde{\omega}^{\dot{\beta}\dot{\gamma}} - \lambda^2 \tilde{\omega}_{\gamma, \dot{\beta}} \tilde{\omega}^{\gamma\dot{\beta}}, \quad (9)$$

where the parameter  $\lambda$  is proportional to the inverse radius of anti-de Sitter space,  $\lambda \sim r^{-1}$ .  $AdS_4$  space is described by the vierbein and connections satisfying the equations

$$\tilde{R}^{\alpha, \dot{\beta}} = 0, \quad \tilde{R}^{\alpha\alpha} = 0, \quad \tilde{R}^{\dot{\beta}\dot{\beta}} = 0. \quad (10)$$

Linearized HS curvatures are

$$\begin{aligned} R_1^{\alpha(m), \dot{\beta}(n)} &= \tilde{D}\omega^{\alpha(m), \dot{\beta}(n)} + \\ &+ n\theta(m-n)\tilde{h}_{\gamma, \dot{\beta}} \omega^{\gamma\alpha(m), \dot{\beta}(n-1)} + n\lambda^2\theta(n-m-2)\tilde{h}_{\gamma, \dot{\beta}} \omega^{\gamma\alpha(m), \dot{\beta}(n-1)} + \\ &+ m\theta(n-m)\tilde{h}^{\alpha, \dot{\delta}} \omega^{\alpha(m-1), \dot{\beta}(n)\dot{\delta}} + m\lambda^2\theta(m-n-2)\tilde{h}^{\alpha, \dot{\delta}} \omega^{\alpha(m-1), \dot{\beta}(n)\dot{\delta}}, \end{aligned} \quad (11)$$

where  $\theta(x)$  is the step-function

$$\theta(x) = \begin{cases} 1 & \text{at } x \geq 0; \\ 0 & \text{at } x < 0. \end{cases} \quad (12)$$

The curvatures (11) satisfy the Bianchi identities [15]

$$\begin{aligned} \tilde{D}R_1^{\alpha(m), \dot{\beta}(n)} = & -\lambda^{(|m-n|/2)+1} (m\lambda^{-|m-n-2|/2} \tilde{h}^{\alpha}_{\dot{\delta}} R_1^{\alpha(m-1), \dot{\beta}(n)\dot{\delta}} + \\ & + n\lambda^{-|m-n+2|/2} \tilde{h}_{\gamma, \dot{\beta}} R_1^{\alpha(m)\gamma, \dot{\beta}(n-1)}). \end{aligned} \quad (13)$$

It is convenient to introduce  $H_{\alpha\beta}$  and  $\bar{H}_{\dot{\alpha}\dot{\beta}}$

$$\tilde{h}_{\alpha, \dot{\beta}} \tilde{h}_{\gamma, \dot{\delta}} = \frac{1}{2} \epsilon_{\alpha\gamma} \bar{H}_{\dot{\beta}\dot{\delta}} + \frac{1}{2} \epsilon_{\dot{\beta}\dot{\delta}} H_{\alpha\gamma}, \quad (14)$$

$$H_{\alpha\beta} = \tilde{h}_{\alpha, \dot{\gamma}} \tilde{h}_{\beta, \dot{\gamma}}, \quad \bar{H}_{\dot{\alpha}\dot{\beta}} = \tilde{h}_{\gamma, \dot{\alpha}} \tilde{h}^{\gamma}_{\dot{\beta}}. \quad (15)$$

Free field equations for massless fields of spins  $s \geq 2$  in Minkowski space can be written in the form [15]

$$R_1^{\alpha(m), \dot{\beta}(n)} = 0 \quad \text{for} \quad n > 0, m > 0, n + m = 2(s - 1); \quad (16)$$

$$R_1^{\alpha(m)} = C^{\alpha(m)\gamma\delta} H_{\gamma\delta} \quad \text{for} \quad m = 2(s - 1); \quad (17)$$

$$R_1^{\dot{\beta}(n)} = \bar{C}^{\dot{\beta}(n)\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}} \quad \text{for} \quad n = 2(s - 1). \quad (18)$$

Equations (16)-(18) are equivalent to the equations of motion which follow from the Fronsdal action [14] supplemented with certain algebraic constraints which express connections  $\omega_{\alpha(m), \dot{\beta}(n)}$  via  $\frac{1}{2}|m-n|$  derivatives of the dynamical frame-like HS field. The multispinor zero-forms  $C^{\alpha(2s)}$  and  $\bar{C}^{\dot{\beta}(2s)}$ , which remain non-zero on-shell, are spin- $s$  analogues of the Weyl tensor in gravity.

HS gauge transformation is

$$\begin{aligned} \delta\omega^{\alpha(m), \dot{\beta}(n)} = & \tilde{D}\epsilon^{\alpha(m), \dot{\beta}(n)} + \\ & + n\theta(m-n) \tilde{h}_{\gamma, \dot{\beta}} \epsilon^{\gamma\alpha(m), \dot{\beta}(n-1)} + n\lambda^2\theta(n-m-2) \tilde{h}_{\gamma, \dot{\beta}} \epsilon^{\gamma\alpha(m), \dot{\beta}(n-1)} + \\ & + m\theta(n-m) \tilde{h}^{\alpha}_{\dot{\delta}} \epsilon^{\alpha(m-1), \dot{\beta}(n)\dot{\delta}} + m\lambda^2\theta(m-n-2) \tilde{h}^{\alpha}_{\dot{\delta}} \epsilon^{\alpha(m-1), \dot{\beta}(n)\dot{\delta}}, \end{aligned} \quad (19)$$

where a gauge parameter  $\epsilon^{\alpha(m), \dot{\beta}(n)}(x)$  is an arbitrary function of  $x$ . Note, that the limit  $\lambda \rightarrow 0$  gives the proper description of HS fields in 4d Minkowski space.

As explained in [10], to obtain currents with odd and even spins, the connections  $\omega^{i; \alpha(m), \dot{\beta}(n)}$  and curvatures  $R^{i; \alpha(m), \dot{\beta}(n)}$  should be endowed with an additional color index  $i = 1 \dots N$  which labels independent dynamical fields. To contract

color indices, we introduce the real tensor  $c_{ijk}$  which can be either symmetric or antisymmetric. Color indices are raised and lowered by the Euclidean metric  $g_{ij}$ , which is convenient to set  $g_{ij} = \delta_{ij}$ .

Free fields are described by the quadratic action [15]

$$S_0 = \frac{1}{2} \int \sum_{m,n \geq 0} \frac{1}{m!n!} \varepsilon(m-n) \lambda^{-|m-n|} R_1^{i;\alpha(m), \dot{\beta}(n)} R_{1\ i;\alpha(m), \dot{\beta}(n)}, \quad (20)$$

where  $\varepsilon(x) = \theta(x) - \theta(-x)$  and  $m+n = 2(s-1)$ ,  $s$  is a spin of a dynamical field.

Following [11, 12], to obtain a cubic deformation of the quadratic action, linear curvature  $R_1$  in the action (20) has to be replaced by  $R = R_1 + R_2$  where

$$\begin{aligned} R_2^{i;\alpha(m), \dot{\beta}(n)} &= \\ &= \sum_{p,q,k,l,u,v \geq 0} \lambda^{1+d_0-d_1-d_2} \frac{m!n!}{p!q!k!l!u!v!} c^i_{j k} \omega^{j;\alpha(p)}_{\gamma(k), \dot{\delta}(l)} \omega^{k;\alpha(q)\gamma(k), \dot{\delta}(l)\dot{\beta}(v)}_{\gamma(k), \dot{\delta}(l)} \omega^{k;\alpha(q)\gamma(k), \dot{\delta}(l)\dot{\beta}(v)}, \end{aligned} \quad (21)$$

$$d_0 = \frac{m-n}{2}, \quad d_1 = \frac{p+k-l-u}{2}, \quad d_2 = \frac{q+k-l-v}{2}.$$

The nonlinear action is

$$S_1 = \frac{1}{2} \int \sum_{m,n \geq 0} \frac{1}{m!n!} \varepsilon(m-n) \lambda^{-|m-n|} R^{i;\alpha(m), \dot{\beta}(n)} R_{i;\alpha(m), \dot{\beta}(n)}. \quad (22)$$

### 3 Problem

It is convenient to describe currents as Hodge dual differential forms. In these terms, the on-shell closure condition for the latter is traded for the current conservation condition. In this paper, we consider spin- $t$  currents built from two HS connections with arbitrary  $s_1, s_2 > 0$  in  $AdS_4$ , where  $t \leq s_1 + s_2 - 1$ . Such currents contain the minimal possible number of derivatives of the dynamical fields. Analogous problem in  $4d$  Minkowski space has been solved in [10] for the case of  $s_1 = s_2$ . Here we derive the form of the currents from the nonlinear action (22).

An arbitrary variation of the action (22) can be represented in the form

$$\delta S_1 = \int \sum_{t,s_1,s_2} \sum_{m,n} J_{t,s_1,s_2}^{i;\alpha(m), \dot{\beta}(n)} \delta \omega_{i;\alpha(m), \dot{\beta}(n)}, \quad (23)$$

where  $m+n = 2(t-1)$ . The current  $J_{t,s_1,s_2}^{i;\alpha(m), \dot{\beta}(n)}$  carries the color index  $i$ . Actually, there are  $N$  copies of a current, one for each value of  $i$ , and we can set

$i = 1$  without loss of generality. In what follows this index  $i = 1$  will be omitted in all current forms. Also, it is convenient to denote  $c_{jk} := c_{1jk}$  with  $c_{jk}$  being either symmetric or antisymmetric.

To define a nontrivial HS charge as an integral over a  $3d$  space, one should find such current three-form  $J_{t,s_1,s_2}(x)$  built from dynamical HS fields that is closed by virtue of HS field equations (16)-(18), but not exact. The conserved current three-form is

$$J_{t,s_1,s_2}(x) = \sum_{m,n} \frac{\lambda^{-|m-n|}}{m!n!} \xi_{\alpha(m),\dot{\beta}(n)}(x) J_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)}(x), \quad m+n=2(t-1), \quad (24)$$

where  $t$  is a spin of the current,  $\xi_{\alpha(m),\dot{\beta}(n)}$  are global symmetry parameters, which can be identified with those gauge symmetry parameters that leave the gauge fields invariant. Factor  $\frac{\lambda^{-|m-n|}}{m!n!}$  is introduced for convenience. These parameters obey the conditions, which follow from (19)

$$D\xi^{\alpha(m),\dot{\beta}(n)} := \tilde{D}\xi^{\alpha(m),\dot{\beta}(n)} + n(\theta(m-n) + \lambda^2\theta(n-m-2))\tilde{h}_{\gamma,\dot{\beta}}^{\dot{\beta}}\xi^{\gamma\alpha(m),\dot{\beta}(n-1)} + \\ + m(\theta(n-m) + \lambda^2\theta(m-n-2))\tilde{h}^{\alpha}_{\dot{\delta}}\xi^{\alpha(m-1),\dot{\beta}(n)\dot{\delta}} = 0. \quad (25)$$

One can see, that

$$dJ_{t,s_1,s_2} = \sum_{m,n} D\xi_{\alpha(m),\dot{\beta}(n)} J_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)} + \sum_{m,n} \xi_{\alpha(m),\dot{\beta}(n)} D J_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)}. \quad (26)$$

Finally, for the parameters obeying (25), the conservation condition is equivalent to a set of equations

$$D J_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)} \simeq 0, \quad m+n=2(t-1). \quad (27)$$

For the currents defined via (23) the conservation condition (27) holds as a consequence of the gauge invariance of the action proven in [11].

Conserved currents generate conserved charges. By the Noether theorem the latter are generators of global symmetries. Hence, one should expect as many conserved charges as global symmetry parameters. For a spin  $t$ , there are as many global symmetry parameters as the gauge parameters  $\epsilon_{\alpha(m),\dot{\beta}(n)}$  with  $m+n=2(t-1)$ .

In what follows we will use notations

$$D^{top}\omega^{\alpha(m),\dot{\beta}(n)} = -n\theta(m-n)\tilde{h}_{\gamma,\dot{\beta}}^{\dot{\beta}}\omega^{\gamma\alpha(m),\dot{\beta}(n-1)} - \\ - m\theta(n-m)\tilde{h}^{\alpha}_{\dot{\delta}}\omega^{\alpha(m-1),\dot{\beta}(n)\dot{\delta}}, \quad (28)$$

$$D^{sub}\omega^{\alpha(m),\dot{\beta}(n)} = -n\theta(n-m-2)\tilde{h}_{\gamma,\dot{\beta}}^{\dot{\beta}}\omega^{\gamma\alpha(m),\dot{\beta}(n-1)} - \\ - m\theta(m-n-2)\tilde{h}^{\alpha}_{\dot{\delta}}\omega^{\alpha(m-1),\dot{\beta}(n)\dot{\delta}}, \quad (29)$$

$$D^{cur}\omega^{\alpha(m),\dot{\beta}(n)} = R_1^{\alpha(m),\dot{\beta}(n)}. \quad (30)$$

From free field equations (16) it follows that

$$D^{cur}\omega^{\alpha(m)},\dot{\beta}(n)} \simeq \delta_{n,0}C^{\alpha(m)\gamma\delta}\tilde{H}_{\gamma\delta} + \delta_{m,0}\bar{C}^{\dot{\beta}(n)\dot{\gamma}\dot{\delta}}\tilde{h}_{\dot{\gamma}\dot{\delta}}. \quad (31)$$

$D^{top}$  and  $D^{sub}$  can act on  $J_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)}(x)$  and  $R_2^{\alpha(m),\dot{\beta}(n)}$  (21) because they are bilinear in connections. Also,  $D^{top}$  and  $D^{sub}$  can act on  $\xi^{\alpha(m),\dot{\beta}(n)}$  the same way as on  $\omega^{\alpha(m),\dot{\beta}(n)}$ . In these terms,

$$\tilde{D}\omega^{\alpha(m),\dot{\beta}(n)} = D^{top}\omega^{\alpha(m),\dot{\beta}(n)} + \lambda^2 D^{sub}\omega^{\alpha(m),\dot{\beta}(n)} + D^{cur}\omega^{\alpha(m),\dot{\beta}(n)}. \quad (32)$$

Note, that the  $\lambda$ -dependent term vanishes in the Minkowski case. It is convenient to denote the flat background part of the covariant derivative  $\tilde{D}$  as  $D^{fl}$

$$D^{fl} := D^{top} + D^{cur}. \quad (33)$$

If the three-form  $J_{t,s_1,s_2}$  obeys (27) on shell, the charge

$$Q_\xi = \int_{M^3} J_{t,s_1,s_2} \quad (34)$$

is conserved by virtue of (25) and (28), (29). As a result, there are as many conserved charges  $Q_\xi$  as independent global symmetry parameters  $\xi$ . Nontrivial charges are represented by the current  $J_{t,s_1,s_2}(x)$  cohomology, *i.e.*, closed currents modulo exact ones  $J_{t,s_1,s_2} \simeq d\Psi_{t,s_1,s_2}$ . Since the currents should be closed on-shell, *i.e.*, by virtue of the free field equations (16)-(18), analysis is greatly simplified by the fact that all linearized HS curvatures  $R_1^{\alpha(m),\dot{\beta}(n)}$  with  $m > 0$ ,  $n > 0$  are zero on shell.

Conservation of currents does not imply that they are invariant under the gauge transformations (19). However, as shown below, the gauge variation of  $J_{t,s_1,s_2}$  is exact

$$\delta J_{t,s_1,s_2}(x) \simeq dH_{t,s_1,s_2}(x), \quad (35)$$

so that the charge  $Q_\xi$  turns out to be gauge invariant.

Thus, the problem is

- (a) to find current three-forms (24) from the variation of action,
- (b) to check that these forms obey the conservation condition (28), (29),
- (c) to check that in the flat limit  $\lambda \rightarrow 0$  these forms give currents from [10],
- (d) to check that these forms are Hermitian to give rise to real charges,
- (e) to check that the HS charges are gauge invariant.

## 4 Variation of the action

The variation of nonlinear curvature  $R^{i;\alpha(m)},\dot{\beta}(n)$  is

$$\delta R^{i;\alpha(m)},\dot{\beta}(n) = \delta R_1^{i;\alpha(m)},\dot{\beta}(n) + \delta R_2^{i;\alpha(m)},\dot{\beta}(n), \quad (36)$$

where

$$\begin{aligned} \delta R_1^{i;\alpha(m)},\dot{\beta}(n) &= \tilde{D}\delta\omega^{i;\alpha(m)},\dot{\beta}(n) + \\ &+ n(\theta(m-n) + \lambda^2\theta(n-m-2)) \tilde{h}_{\gamma,\dot{\beta}} \delta\omega^{i;\gamma\alpha(m)},\dot{\beta}(n-1) + \\ &+ m(\theta(n-m) + \lambda^2\theta(m-n-2)) \tilde{h}_{\gamma,\dot{\delta}}^\alpha \delta\omega^{i;\alpha(m-1)},\dot{\beta}(n)\dot{\delta} \end{aligned} \quad (37)$$

and

$$\begin{aligned} \delta R_2^{i;\alpha(m)},\dot{\beta}(n) &= \\ &= 2 \sum_{p,q,k,l,u,v} \lambda^{1+d_0-d_1-d_2} \frac{m!n!}{p!q!k!l!u!v!} c^i_{jk} \omega^{j;\alpha(p)}_{\gamma(k),\dot{\delta}(l)} \delta\omega^{k;\alpha(q)\gamma(k)},\dot{\delta}(l)\dot{\beta}(v). \end{aligned} \quad (38)$$

The variation of action (22) is

$$\begin{aligned} \delta S_1 &= \int \sum_{m,n} \varepsilon(m-n) \frac{\lambda^{-|m-n|}}{m!n!} R^{i;\alpha(m)},\dot{\beta}(n) \delta R_{i;\alpha(m),\dot{\beta}(n)} = \\ &= \int \sum_{m,n} \varepsilon(m-n) \frac{\lambda^{-|m-n|}}{m!n!} (R_1^{i;\alpha(m)},\dot{\beta}(n) \delta R_{i;\alpha(m),\dot{\beta}(n)} + R_2^{i;\alpha(m)},\dot{\beta}(n) \delta R_{i;\alpha(m),\dot{\beta}(n)} + \\ &\quad + R_1^{i;\alpha(m)},\dot{\beta}(n) \delta R_{2\ i;\alpha(m),\dot{\beta}(n)} + R_2^{i;\alpha(m)},\dot{\beta}(n) \delta R_{2\ i;\alpha(m),\dot{\beta}(n)}). \end{aligned} \quad (39)$$

The first term is the variation of the action  $S_0$  (20) which vanishes on equations of motion (16) - (18). The last term is cubic in connections  $\omega^{i;\alpha(m)},\dot{\beta}(n)$  and does not contribute to bilinear currents. The second and third terms give rise to the currents. Using (11), (17), (18), (32), (37), (38) and integrating by parts we obtain

$$\begin{aligned} \delta S_1 &\simeq \int \sum_{m,n} \varepsilon(m-n) \frac{\lambda^{-|m-n|}}{m!n!} [-(D^{top} + \lambda^2 D^{sub} + D^{cur}) R_2^{i;\alpha(m),\dot{\beta}(n)} \delta\omega_{i;\alpha(m),\dot{\beta}(n)} - \\ &- n(\theta(m-n) + \lambda^2\theta(n-m-2)) R_2^{i;\alpha(m),\dot{\theta}\dot{\beta}(n-1)} \tilde{h}_{\gamma,\dot{\theta}}^\gamma \delta\omega_{i;\gamma\alpha(m),\dot{\beta}(n-1)} + \\ &+ m(\theta(n-m) + \lambda^2\theta(m-n-2)) R_2^{i;\alpha(m-1)\gamma,\dot{\beta}(n)} \tilde{h}_{\gamma,\dot{\delta}}^\delta \delta\omega_{i;\alpha(m-1),\dot{\beta}(n)\dot{\delta}}] + \\ &+ \int \sum_{r>0} \frac{\lambda^{-r}}{r!} (C^{i;\alpha(r)\gamma\delta} H_{\gamma\delta} \delta R_{2\ i;\alpha(r)} - \bar{C}^{i;\dot{\beta}(r)\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}} \delta R_{2\ i;\dot{\beta}(r)}). \end{aligned} \quad (40)$$



Finally, currents can be written down (the color index  $i = 1$  is omitted) directly from (40) for  $t > 1$

$$J_{t,s_1,s_2} = \sum_{m,n} \xi_{\alpha(m),\dot{\beta}(n)} \frac{\delta S_1}{\delta \omega_{\alpha(m),\dot{\beta}(n)}}, \quad m+n = 2(t-1). \quad (41)$$

## 5 Examples

### 5.1 Spin-two current built from fields of equal spins

Consider the case of a current with  $t = 2$ ,  $s_1 = s_2 = s > 1$  to illustrate the structure of the current three-form and analyze the flat limit  $\lambda \rightarrow 0$ . The current is

$$J_{2,s} = \frac{\lambda^{-2}}{2} \xi_{\alpha\alpha} J_{2,s}^{\alpha\alpha} + \xi_{\alpha,\dot{\beta}} J_{2,s}^{\alpha,\dot{\beta}} + \frac{\lambda^{-2}}{2} \xi_{\dot{\beta}\dot{\beta}} J_{2,s}^{\dot{\beta}\dot{\beta}}. \quad (42)$$

where we denote  $J_{2,s} := J_{2,s,s}$ . Using (21), (30), (31), (38) we obtain

$$J_{2,s}^{\alpha\alpha} = -(D^{top} + \lambda^2 D^{sub}) \sum_{m,n} \frac{2\lambda^{2-|m-n|}}{(m-1)!n!} c_{ij} \omega^{i;\alpha\gamma(m-1),\dot{\delta}(n)} \omega^{j;\alpha}_{\gamma(m-1),\dot{\delta}(n)}, \quad (43)$$

$$\begin{aligned} J_{2,s}^{\alpha,\dot{\beta}} = & \sum_{m,n} 2\lambda^{2-|m-n|} \left[ \frac{1}{(m-1)!n!} c_{ij} \omega^{i;\alpha\gamma(m-1),\dot{\delta}(n)} \omega^{j;\varphi}_{\gamma(m-1),\dot{\delta}(n)} \tilde{h}_{\varphi}^{\dot{\beta}} - \right. \\ & \left. - \frac{1}{m!(n-1)!} c_{ij} \omega^{i;\gamma(m),\dot{\delta}(n-1)\dot{\theta}} \omega^{j;}_{\gamma(m),\dot{\delta}(n-1)} \tilde{h}^{\dot{\beta}}_{\dot{\theta}} \right] + \\ & + \frac{2\lambda^{4-2s}}{(2s-3)!} [c_{ij} C^{i;\alpha\gamma(2s-3)\varphi\rho} H_{\varphi\rho} \omega_{\gamma(2s-3)}^{\dot{\beta}} - c_{ij} \bar{C}^{i;\dot{\delta}(2s-3)\dot{\beta}\psi\dot{\theta}} \bar{H}_{\psi\dot{\theta}} \omega_{\dot{\delta}(2s-3)}^{\alpha}], \quad (44) \end{aligned}$$

$$J_{2,s}^{\dot{\beta}\dot{\beta}} = -(D^{top} + \lambda^2 D^{sub}) \sum_{m,n} \frac{2\lambda^{2-|m-n|}}{m!(n-1)!} c_{ij} \omega^{i;\gamma(m),\dot{\delta}(n-1)\dot{\beta}} \omega^{j;}_{\gamma(m),\dot{\delta}(n-1)} \dot{\beta}. \quad (45)$$

Note, that  $m+n = 2(s-1)$ ,  $m, n \geq 0$ .

There are terms with various powers of  $\lambda$  and  $C$ -dependent terms in (43), (44), (45). The parts of the currents containing inverse powers of  $\Lambda$  contain the higher derivatives. To obtain a correct  $\lambda \rightarrow 0$  limit one should cancel out these terms

with an exact form  $d\Psi_{2,s}$ , where

$$\begin{aligned} \Psi_{2,s} = & \frac{2\lambda^{4-2(s-m)}}{m!(2s-3-m)!} \sum_{m=0}^{s-3} [\xi_{\alpha\alpha} c_{ij} \omega^{i;\alpha\gamma(m)}, \dot{\delta}(2s-3-m) \omega^{i;\alpha}{}_{\gamma(m),\dot{\delta}(2s-3-m)} + \\ & + \xi_{\alpha,\dot{\beta}} (c_{ij} \omega^{i;\alpha\gamma(2s-3-m)}, \dot{\delta}(m) \omega^{j;\gamma(2s-3-m),\dot{\delta}(m)} \dot{\beta} - c_{ij} \omega^{i;\alpha\gamma(m)}, \dot{\delta}(2s-3-m) \omega^{j;\gamma(m),\dot{\delta}(2s-3-m)} \dot{\beta}) - \\ & - \xi_{\dot{\beta}\dot{\beta}} c_{ij} \omega^{i;\alpha\gamma(2s-3-m)}, \dot{\delta}(m) \dot{\beta} \omega^{i;\gamma(2s-3-m),\dot{\delta}(m)} \dot{\beta}] . \quad (46) \end{aligned}$$

In the case of  $s = 2$  there is no need in adding an exact form.

The fact that complete antisymmetrization over a group of three dotted (undotted) indices gives zero gives the relation

$$c_{ij} \omega^{i;\alpha\gamma(m-1)}, \dot{\delta}(m) \omega^{j;\varphi}{}_{\gamma(m-1),\dot{\delta}(m)} \tilde{h}_\varphi{}^{\dot{\beta}} = c_{ij} \omega^{i;\gamma(m)}, \dot{\delta}(m-1) \dot{\beta} \omega^{j;\gamma(m),\dot{\delta}(m-1)} \dot{\theta} \tilde{h}^\alpha{}_{\dot{\theta}}, \quad (47)$$

to be used in the analysis.

Straightforward calculation gives

$$\hat{J}_{2,s} = J_{2,s} + d\Psi_{2,s} = \frac{\lambda^{-2}}{2} \xi_{\alpha\alpha} \hat{J}_{2,s}{}^{\alpha\alpha} + \xi_{\alpha,\dot{\beta}} \hat{J}_{2,s}{}^{\alpha,\dot{\beta}} + \frac{\lambda^{-2}}{2} \xi_{\dot{\beta}\dot{\beta}} \hat{J}_{2,s}{}^{\dot{\beta}\dot{\beta}}, \quad (48)$$

where

$$\begin{aligned} \hat{J}_{2,s}{}^{\alpha\alpha} = & 2\lambda^2 [2c_{ij} \omega^{i;\alpha\varphi\gamma(s-2)}, \dot{\delta}(s-2) \omega^{j;\alpha}{}_{\gamma(s-2),\dot{\delta}(s-2)} \tilde{h}_\varphi{}^{\dot{\theta}} + \\ & + \frac{1}{2(s-1)} D^{fl} c_{ij} \omega^{i;\alpha\gamma(s-2)}, \dot{\delta}(s-1) \omega^{j;\alpha}{}_{\gamma(s-2),\dot{\delta}(s-1)} + \\ & + \frac{1}{s-1} (c_{ij} \omega^{i;\alpha\gamma(s-2)}, \dot{\delta}(s-1) \omega^{j;\gamma(s-2),\dot{\delta}(s-1)} \dot{\theta} - \\ & - c_{ij} \omega^{i;\alpha\gamma(s-1)}, \dot{\delta}(s-2) \omega^{j;\gamma(s-1),\dot{\delta}(s-2)} \dot{\theta}) \tilde{h}^\alpha{}_{\dot{\theta}}], \quad (49) \end{aligned}$$

$$\begin{aligned} \hat{J}_{2,s}{}^{\alpha,\dot{\beta}} = & 2c_{ij} \omega^{i;\alpha\varphi\gamma(s-2)}, \dot{\delta}(s-2) \omega^{j;\gamma(s-2),\dot{\delta}(s-2)} \dot{\theta} \tilde{h}_\varphi{}^{\dot{\beta}} + \\ & + \frac{1}{s-1} D^{fl} (c_{ij} \omega^{i;\alpha\gamma(s-2)}, \dot{\delta}(s-1) \omega^{j;\gamma(s-2),\dot{\delta}(s-1)} \dot{\beta} - \\ & - c_{ij} \omega^{i;\alpha\gamma(s-1)}, \dot{\delta}(s-2) \omega^{j;\gamma(s-1),\dot{\delta}(s-2)} \dot{\beta}), \quad (50) \end{aligned}$$

$$\begin{aligned} \hat{J}_{2,s}{}^{\dot{\beta}\dot{\beta}} = & 2\lambda^2 [2c_{ij} \omega^{i;\varphi\gamma(s-2)}, \dot{\delta}(s-2) \dot{\beta} \omega^{j;\gamma(s-2),\dot{\delta}(s-2)} \dot{\theta} \tilde{h}_\varphi{}^{\dot{\beta}} + \\ & + \frac{1}{2(s-1)} D^{fl} c_{ij} \omega^{i;\gamma(s-1)}, \dot{\delta}(s-2) \dot{\beta} \omega^{j;\gamma(s-1),\dot{\delta}(s-2)} \dot{\beta} + \\ & + \frac{1}{s-1} (c_{ij} \omega^{i;\varphi\gamma(s-2)}, \dot{\delta}(s-1) \omega^{j;\gamma(s-2),\dot{\delta}(s-1)} \dot{\beta} - \\ & - c_{ij} \omega^{i;\varphi\gamma(s-1)}, \dot{\delta}(s-2) \omega^{j;\gamma(s-1),\dot{\delta}(s-2)} \dot{\beta}) \tilde{h}_\varphi{}^{\dot{\beta}}]. \quad (51) \end{aligned}$$

$\hat{J}_{2,s}$  is  $\lambda$ -independent. One can check that the form  $\hat{J}_{2,s}$  is closed and obeys

$$D\hat{J}_{2,s}^{\alpha\alpha} = 2\lambda^2 \tilde{h}^{\alpha, \dot{\theta}}_{\dot{\theta}} \hat{J}_{2,s}^{\alpha, \dot{\theta}} + (D^{top} + \lambda^2 D^{sub}) \hat{J}_{2,s}^{\alpha\alpha} \simeq 0, \quad (52)$$

$$D\hat{J}_{2,s}^{\alpha, \dot{\beta}} = \tilde{h}^{\alpha, \dot{\theta}}_{\dot{\theta}} \hat{J}_{2,s}^{\dot{\theta}\dot{\beta}} + \tilde{h}_{\gamma, \dot{\beta}}^{\dot{\beta}} \hat{J}_{2,s}^{\alpha\gamma} + (D^{top} + \lambda^2 D^{sub}) \hat{J}_{2,s}^{\alpha, \dot{\beta}} \simeq 0, \quad (53)$$

$$D\hat{J}_{2,s}^{\dot{\beta}\dot{\beta}} = 2\lambda^2 \tilde{h}_{\gamma, \dot{\beta}}^{\dot{\beta}} \hat{J}_{2,s}^{\gamma, \dot{\beta}} + (D^{top} + \lambda^2 D^{sub}) \hat{J}_{2,s}^{\dot{\beta}\dot{\beta}} \simeq 0. \quad (54)$$

From [10] we know the spin-two current  $J_{2,s}^M$  in Minkowski space. Since the  $AdS_4$  current  $\hat{J}_{2,s}$  (48) has  $\lambda$ -independent form, it must have the same form in flat limit  $\lambda \rightarrow 0$ . From (49) - (51) we have the relation

$$J_{2,s}^M + Q^{fl} \chi_{2,s} = \hat{J}_{2,s}, \quad (55)$$

where

$$\begin{aligned} \chi_{2,s} = & \frac{\lambda^2}{s-1} \xi_{\alpha\alpha} c_{ij} \omega^{i; \alpha\gamma(s-2), \dot{\delta}(s-1)} \omega^{j; \alpha}_{\gamma(s-2), \dot{\delta}(s-1)} + \\ & + \frac{1}{s-1} \xi_{\alpha\dot{\beta}} (c_{ij} \omega^{i; \alpha\gamma(s-2), \dot{\delta}(s-1)} \omega^{j; \gamma(s-2), \dot{\delta}(s-1)}{}^{\dot{\beta}} - c_{ij} \omega^{i; \alpha\gamma(s-1), \dot{\delta}(s-2)} \omega^{j; \gamma(s-1), \dot{\delta}(s-2)}{}^{\dot{\beta}}) + \\ & + \frac{\lambda^2}{s-1} c_{ij} \omega^{i; \gamma(s-1), \dot{\delta}(s-2)}{}^{\dot{\beta}} \omega^{j; \gamma(s-1), \dot{\delta}(s-2)}{}^{\dot{\beta}}, \end{aligned} \quad (56)$$

which proves the flat limit of the current (48) reproduces the previous results of [10]. One can check that the current is Hermitian. Note, that this is only true if  $c_{ij}$  is symmetric.

## 5.2 Spin-one current

Since the action (22) has no dynamical term for spin-one field  $\omega^i$  (with no spinor indices) following [11] it should be added to the action in a standard way

$$S_{EM} = \int R_i^* R^i, \quad (57)$$

where  $*$  is the Hodge star operator, and, as it follows from (11), (21)

$$R^i = d\omega^i + \sum_{k,l \geq 0} \frac{\lambda^{1-|m-n|}}{k!l!} c^i_{jk} \omega^{j; \gamma(k), \dot{\delta}(l)} \omega^{k; \gamma(k), \dot{\delta}(l)}. \quad (58)$$

Neglecting the on-shell trivial and quartic terms, the variation of the full action  $S_{full} = S_1 + S_{EM}$  is

$$\delta S_{full} = \delta S_1 + \int [R_{1i}^* \delta R_2^i + R_{2i}^* \delta R_1^i]. \quad (59)$$

Note that the form of spin-one equation is slightly different from (16)-(18)

$$R_1^i = C^{i;\gamma\delta} H_{\gamma\delta} + \bar{C}^{i;\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}}. \quad (60)$$

From the properties of Pauli matrices it follows that

$$R_1^{i*} = i(C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}}). \quad (61)$$

Consider the case of  $t = 1 = s_1 = s_2 = 1$ . Integrating by parts, one can write down the corresponding current  $J_{1,1}$  from (59) as a coefficient of  $\delta\omega^i$  (color index  $i$  is omitted)

$$J_{1,1} = 2\xi \lambda i c_{ij} (C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}}) \omega^j - d(\xi^* R_2^i), \quad (62)$$

where  $\xi$  is a global symmetry parameter zero-form (25) with no spinor indices. One can transform  $J_{1,1}$  to get

$$\hat{J}_{1,1} = \frac{1}{\lambda} (J_{1,1} + d(\xi^* R_2^i)) = 2\xi i c_{ij} (C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}}) \omega^j. \quad (63)$$

One can check, that the  $C$ -dependent terms are not exact. The current (63) coincides with Minkowski current  $J_{1,1}^M$  from [10] modulo an overall factor of 2.

Consider the case of  $t = 1, s_1 = s_2 = s > 1$ . The current follows from the  $C$ -dependent terms of (40) with the aid of (38)

$$J_{1,s} = 2\lambda^{3-2s} \xi c_{ij} (C^{i;\alpha(2s-2)\varphi\rho} H_{\varphi\rho} \omega^{j;\alpha(2s-2)} - \bar{C}^{i;\dot{\beta}(2s-2)\dot{\gamma}\dot{\delta}} \bar{H}_{\dot{\gamma}\dot{\delta}} \omega^{j;\dot{\beta}(2s-2)}). \quad (64)$$

Note, that there are no currents with  $t = 1, s_1 \neq s_2$ . Also, note that equation (63) follows from (64) at  $s = 1$ . The current three-form  $J_{1,s}$  (64) is on-shell closed only if  $c_{ij}$  is antisymmetric.

For  $s > 1$   $J_{1,s}$  (64) can be rewritten in the form bilinear in connections by adding an exact form

$$\begin{aligned} \Psi_{1,s} = 2\xi \lambda^{3-2s} \sum_{m=0}^{s-2} (-1)^{k+1} 2^k \lambda^{2m} \frac{(s-1)!}{(s-k-1)!} c_{ij} & (\omega^{i;\alpha(2s-2-m)}, \dot{\beta}(m) \omega^{j;\alpha(2s-2-m),\dot{\beta}(m)} - \\ & - \omega^{i;\alpha(m)}, \dot{\beta}(2s-2-m) \omega^{j;\alpha(m),\dot{\beta}(2s-2-m)}) . \end{aligned} \quad (65)$$

As a result,

$$\begin{aligned} \hat{J}_{1,s} &= -\frac{1}{\lambda(-2)^{s-1}s(s-1)!} (J_{1,s} + d\Psi_{1,s}) = \\ &= \xi c_{ij} [\omega^{i;\varphi\gamma(s-2)}, \dot{\delta}(s-1) \omega^{j;\gamma(s-2),\dot{\delta}(s-1)\dot{\theta}} + \omega^{i;\varphi\gamma(s-1)}, \dot{\delta}(s-2) \omega^{j;\gamma(s-1),\dot{\delta}(s-2)\dot{\theta}}] \tilde{h}_{\varphi}, \dot{\theta}. \end{aligned} \quad (66)$$

This three-form is  $\lambda$ -independent, on-shell-closed, Hermitian and reproduces the result of [10]. Thus it is not exact.

## 6 General solution

The  $AdS_4$  conserved currents  $J_{t,s_1,s_2}$  with  $1 < t \leq s_1 + s_2 - 1$  (for definiteness we set  $s_1 \geq s_2$ ) result from the variation of the action (40)

$$\begin{aligned}
J_{t,s_1,s_2} = & \sum_{m,n} \varepsilon(m-n) \frac{\lambda^{-|m-n|}}{m!n!} [-\xi_{\alpha(m),\dot{\beta}(n)} (D^{top} + \lambda^2 D^{sub}) R_2^{\alpha(m),\dot{\beta}(n)}|_{s_1,s_2} - \\
& - n(\theta(m-n) + \lambda^2 \theta(n-m-2)) \xi_{\alpha(m+1),\dot{\beta}(n-1)} R_2^{\alpha(m),\dot{\beta}(n-1)}|_{s_1,s_2} \tilde{h}^\alpha_{,\dot{\theta}} + \\
& + m(\theta(n-m) + \lambda^2 \theta(m-n-2)) \xi_{\alpha(m-1),\dot{\beta}(n+1)} R_2^{\alpha(m-1),\dot{\beta}(n)}|_{s_1,s_2} \tilde{h}^\alpha_{,\dot{\gamma}}] + \\
& + \sum_{p,q,k} \frac{2\lambda^{1-|q+k-t+1|}}{p!q!k!(t-1)!} \delta_{p+q,t-1} \delta_{p+k,2(s_1-1)} \delta_{q+k+t-1,2(s_2-1)} c_{ij} \xi_{\alpha(p+q),\dot{\beta}(p+q)} \times \\
& \times [C^{i;\alpha(p)\gamma(k)\varphi\rho} H_{\varphi\rho} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\beta}(t-1)} - \bar{C}^{i;\dot{\beta}(p)\dot{\delta}(k)\dot{\varphi}\dot{\rho}} \bar{H}_{\dot{\varphi}\dot{\rho}} \omega^{j;\alpha(t-1)}_{\gamma(k),\dot{\delta}(k)} \dot{\beta}(q)], \quad (67)
\end{aligned}$$

where  $R_2^{\alpha(m),\dot{\beta}(n)}|_{s_1,s_2}$  is the restriction of (21) to terms containing connections with spins  $s_1$  and  $s_2$ . The current (67) contains  $s_1 + s_2 - 2$  derivatives of the frame field.

To check nonexactness of the three-form (67) it suffices to add an exact form

$$d\Psi_{t,s_1,s_2} = d\left(\sum_{m,n} \xi_{\alpha(m),\dot{\beta}(n)} \Psi_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)}\right), \quad n+m=2(t-1),$$

where

$$\begin{aligned}
\Psi_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)} = & \\
= & \sum_{p,q,k,l,u,v} \frac{2\lambda^{1-\frac{|p+k-l-u|-|q+k-l-v|}{2}}}{p!q!k!l!u!v!} \delta_{p+q,m} \delta_{u+v,n} \delta_{p+k+l+u,2(s_1-1)} \delta_{q+k+l+v,2(s_2-1)} \times \\
& \times \theta(l+u-p-k-1) c_{ij} [\theta(m-n) \omega^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)}_{\gamma(k),\dot{\delta}(l)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) - \\
& - \theta(n-m) \omega^{i;\alpha(u)\gamma(l),\dot{\delta}(k)\dot{\beta}(p)}_{\gamma(l),\dot{\delta}(k)} \omega^{j;\alpha(v)}_{\gamma(l),\dot{\delta}(k)} \dot{\beta}(q)],
\end{aligned}$$

to cancel out  $C$ -dependent terms and make the current three-form  $\lambda$ -independent and containing minimal number of derivatives as in the case of (64). Then one can check nonexactness of the current 3-form  $\hat{J}_{t,s_1,s_2} = J_{t,s_1,s_2} + d\Psi_{t,s_1,s_2}$  in the flat limit  $\lambda \rightarrow 0$  just as in [10].

The current three-form  $\hat{J}_{t,s_1,s_2}$  is

$$\begin{aligned}
\hat{J}_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)} &= \\
&= \sum_{p,q,k,l,u,v} \frac{2\lambda^{1-\frac{|p+k-l-u|-|q+k-l-v|}{2}}}{p!q!k!l!u!v!} \delta_{p+q,m} \delta_{u+v,n} \delta_{p+k+l+u,2(s_1-1)} \delta_{q+k+l+v,2(s_2-1)} c_{ij} \times \\
&\times [\theta(p+k-l-u-1)(D^{top} + \lambda^2 D^{sub})(\theta(m-n)\omega^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) - \\
&\quad - \theta(n-m)\omega^{i;\alpha(u)\gamma(l),\dot{\delta}(k)\dot{\beta}(p)} \omega^{j;\alpha(v)}_{\gamma(l),\dot{\delta}(k)} \dot{\beta}(q)) - \\
&- n(\theta(m-n) + \lambda^2 \theta(n-m-2))((u+1)\omega^{i;\alpha(p-1)\gamma(k),\dot{\delta}(l)\dot{\theta}\dot{\beta}(u)} \omega^{j;\alpha(v)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + (v+1)\omega^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q-1)}_{\gamma(k),\dot{\delta}(l)} \dot{\theta}\dot{\beta}(v) \tilde{h}^\alpha_{\varphi,\dot{\theta}}) + \\
&+ m(\theta(n-m) + \lambda^2 \theta(m-n-2))((p+1)\omega^{i;\alpha(p)\varphi\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) \tilde{h}^\alpha_{\varphi,\dot{\beta}} + \\
&\quad + (q+1)\omega^{i;\alpha(p)\gamma(l),\dot{\delta}(k)\dot{\beta}(u-1)} \omega^{j;\alpha(q)\varphi}_{\gamma(l),\dot{\delta}(k)} \dot{\beta}(v-1) \tilde{h}^\alpha_{\varphi,\dot{\beta}})].
\end{aligned}$$

This current contains  $t - |s_1 - s_2|$  derivatives. This is the minimal possible number of derivatives in the current.

For example, in the case of  $s_1 = s_2 = s$

$$\hat{J}_{t,s} = \sum_{n,m} \frac{\lambda^{-|m-n|}}{m!n!} \xi_{\alpha(m),\dot{\beta}(n)} \hat{J}_{t,s}^{\alpha(m),\dot{\beta}(n)}, \quad n+m=2(t-1),$$

where

$$\begin{aligned}
\hat{J}_{t,s}^{\alpha(t-1),\dot{\beta}(t-1)} &= (t-1)!(t-1)! c_{ij} [\omega^{i;\alpha(t-1)\varphi\gamma(s-2),\dot{\delta}(s-t)} \omega^{j;}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}} \dot{\beta}(t-1) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + \omega^{i;\alpha(t-1)\varphi\gamma(s-t),\dot{\delta}(s-2)} \omega^{j;}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}} \dot{\beta}(t-1) \tilde{h}^\alpha_{\varphi,\dot{\theta}}],
\end{aligned}$$

$$\begin{aligned}
\hat{J}_{t,s}^{\alpha(m),\dot{\beta}(n)} &= \\
&= \lambda^{|m-n|} m!n! \theta(n-m-4) \hat{g}(n) c_{ij} \omega^{i;\alpha(m)\varphi\gamma(s-2),\dot{\delta}(s-t)\dot{\beta}(n-t+1)} \omega^{j;}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}} \dot{\beta}(t-1) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + \delta_{n,t} c_{ij} [2(t-1) \omega^{i;\alpha(m)\varphi\gamma(s-2),\dot{\delta}(s-t)\dot{\beta}} \omega^{j;}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}} \dot{\beta}(n-1) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + \sum_{p=1}^{t-2} \hat{f}(p) \omega^{i;\alpha(m)\varphi\gamma(s-p-1),\dot{\delta}(s-t+p)} \omega^{j;}_{\gamma(s-p-1),\dot{\delta}(s-t+p)} \dot{\beta}(n-1) \tilde{h}^\alpha_{\varphi,\dot{\beta}}] + \\
&+ \lambda^{|m-n|} m!n! \theta(m-n-4) \hat{g}(m) c_{ij} \omega^{i;\alpha(t-1)\varphi\gamma(s-t),\dot{\delta}(s-2)} \omega^{j;\alpha(m-t+1)}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}} \dot{\beta}(n) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + \delta_{m,t} c_{ij} [2(t-1) \omega^{i;\alpha(m-1)\varphi\gamma(s-t),\dot{\delta}(s-2)} \omega^{j;\alpha}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}} \dot{\beta}(n) \tilde{h}^\alpha_{\varphi,\dot{\theta}} + \\
&\quad + \sum_{p=1}^{t-2} \hat{f}(p) \omega^{i;\alpha(m-1)\gamma(s-t+p),\dot{\delta}(s-p-1)} \omega^{j;}_{\gamma(s-t+p),\dot{\delta}(s-p-1)} \dot{\theta}\dot{\beta}(n) \tilde{h}^\alpha_{\varphi,\dot{\theta}}],
\end{aligned}$$

where

$$\begin{aligned}\hat{g}(m) &= \frac{2(t-1)!}{(2t-m-2)!(m-t+1)!}, \quad m \geq t+1, \\ \hat{f}(1) &= \frac{t-1}{s-t+1}, \quad \hat{f}(p) = (t-1) \frac{(s-t)!(s-p)!}{(s-3)!(s-t+p)!}, \quad p > 1.\end{aligned}$$

One can check, that in the case of  $s_1 = s_2$  the current (67) reproduces the one from [10] up to a  $D^{fl}$ -exact form

$$d\chi_{t,s} = D^{fl} \sum_{m,n} \xi_{\alpha(m)\dot{\beta}(n)} \chi_{t,s}^{\alpha(m),\dot{\beta}(n)}, \quad n+m = 2(t-1),$$

where

$$\begin{aligned}\chi_{t,s}^{\alpha(t-1), \dot{\beta}(t-1)} &= f c_{ij} \sum_{p=0}^{\lfloor \frac{t}{2} \rfloor} [\omega^{i;\alpha(t-p-1)\gamma(s-2), \dot{\delta}(s-t+1)\dot{\beta}(p)} \omega^{j;\alpha(p)}_{\gamma(s-2), \dot{\delta}(s-t+1)}^{\dot{\beta}(s-t-p)} + \\ &\quad + \omega^{i;\alpha(t-p-1)\gamma(s-1), \dot{\delta}(s-t)\dot{\beta}(p)} \omega^{j;\alpha(p)}_{\gamma(s-1), \dot{\delta}(s-t)}^{\dot{\beta}(s-t-p)}], \\ \chi_{t,s}^{\alpha(m), \dot{\beta}(n)} &= \\ &= \theta(n-m-2) D^{fl} g(n) c_{ij} \sum_{p=1}^{m+1} \omega^{i;\alpha(m)\gamma(s-p), \dot{\delta}(s-t+p-1)\dot{\beta}(n-t+1)} \omega^{j;\alpha(p)}_{\gamma(s-p), \dot{\delta}(s-t+p-1)}^{\dot{\beta}(t-1)} + \\ &+ \theta(m-n-2) D^{fl} g(m) c_{ij} \sum_{p=1}^{n+1} \omega^{i;\alpha(t-1)\gamma(s-t+p-1), \dot{\delta}(s-p)} \omega^{j;\alpha(m-t+1)}_{\gamma(s-t+p-1), \dot{\delta}(s-p)}^{\dot{\beta}(n)},\end{aligned}$$

where

$$f = \frac{1}{s-t+1}, \quad g(m) = \frac{t-m}{s-p}.$$

It is important, that the conserved currents exist only if  $c_{ij}$  is antisymmetric for odd sum  $t + s_1 + s_2$  and symmetric for even.

Thus, the Hermitian current three-form  $J_{t,s_1,s_2}$  is on-shell closed, but not exact in  $AdS_4$ . It generates the corresponding real conserved charge  $Q = \int J_{t,s_1,s_2}$  that contains as many symmetry parameters as local HS gauge symmetries.

## 7 Gauge transformations

The current three-form (67) is not invariant under the gauge transformations (19). But its gauge variation is an exact form. It is convenient to explain this fact

schematically. Consider a gauge variation of a single term from (67)

$$\begin{aligned}
\delta(\xi\omega_1\omega_2h) &= \xi\omega_1(\tilde{D}\varepsilon_2 - D^{top}\varepsilon_2 - \lambda^2 D^{sub}\varepsilon_2)h + (\tilde{D}\varepsilon_1 - D^{top}\varepsilon_1 - \lambda^2 D^{sub}\varepsilon_1)\omega_2h = \\
&= -\tilde{D}(\xi\omega_1\varepsilon_2h - \xi\varepsilon_1\omega_2h) + \tilde{D}\xi\omega_1\varepsilon_2h + \xi\tilde{D}\omega_1\varepsilon_2h + \tilde{D}\xi\varepsilon_1\omega_2h + \xi\varepsilon_1\tilde{D}\omega_2h - \\
&- \xi\omega_1(D^{top} + \lambda^2 D^{sub})\varepsilon_2h - \xi(D^{top} + \lambda^2 D^{sub})\varepsilon_1\omega_2h = -\tilde{D}(\xi\omega_1\varepsilon_2h - \xi\varepsilon_1\omega_2h) + \\
&+ (D^{top} + \lambda^2 D^{sub})\xi\omega_1\varepsilon_2h + \xi D^{cur}\omega_1\varepsilon_2h + \xi(D^{top} + \lambda^2 D^{sub})\omega_1\varepsilon_2h + \\
&+ (D^{top} + \lambda^2 D^{sub})\xi\varepsilon_1\omega_2h + \xi\varepsilon_1 D^{cur}\omega_2h + \xi\varepsilon_1(D^{top} + \lambda^2 D^{sub})\omega_2h - \\
&- \xi\omega_1(D^{top} + \lambda^2 D^{sub})\varepsilon_2h - \xi(D^{top} + \lambda^2 D^{sub})\varepsilon_1\omega_2h. \quad (68)
\end{aligned}$$

All terms with  $D^{cur}$  are canceled out by  $\delta(\xi CH\omega h)$ . Terms with  $(D^{top} + \lambda^2 D^{sub})$  vanish by the current conservation condition (27), where terms  $\xi\omega_1\omega_2h$  are replaced with  $\xi(\omega_1\varepsilon_2 + \varepsilon_1\omega_2)h$ .

As a result,

$$\delta J_{t,s_1,s_2} \simeq d \sum_{n,m} \xi_{\alpha(m),\dot{\beta}(n)} H_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)},$$

where

$$\begin{aligned}
H_{t,s_1,s_2} &= \\
&- \sum_{m,n} \varepsilon(m-n) \sum_{p,q,k,l,u,v} \delta_{p+q,m} \delta_{u+v,n} \delta_{p+k+l+u,2(s_1-1)} \delta_{q+k+l+v,2(s_2-1)} c_{ij} \xi_{\alpha(p+q),\dot{\beta}(u+v)} \times \\
&\times \frac{\lambda^{1-\frac{|m-n|}{2}-\frac{|p+k-l-u|}{2}-\frac{|q+k-l-v|}{2}}}{p!q!k!l!u!v!} [(D^{top} + \lambda^2 D^{sub}) \varepsilon^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) - \\
&- n(\theta(m-n) + \lambda^2 \theta(n-m-2)) \xi_{\alpha(p+q+1),\dot{\beta}(u+v-1)} \times \\
&\times (\varepsilon^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\theta}\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) \tilde{h}^\alpha_{,\dot{\theta}} + \varepsilon^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\theta}\dot{\beta}(v) \tilde{h}^\alpha_{,\dot{\theta}}) \\
&- m(\theta(n-m) + \lambda^2 \theta(m-n-2)) \xi_{\alpha(p+q-1),\dot{\beta}(u+v+1)} \times \\
&\times (\varepsilon^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) \tilde{h}^\alpha_{,\dot{\theta}} + \varepsilon^{i;\alpha(p)\gamma(k),\dot{\delta}(l)\dot{\beta}(u)} \omega^{j;\alpha(q)\varphi}_{\gamma(k),\dot{\delta}(l)} \dot{\beta}(v) \tilde{h}^\alpha_{,\dot{\theta}})] + \\
&+ 2 \sum_{p,q,k} \frac{\lambda^{2-s_1-\frac{|k-p|}{2}}}{p!q!k!(p+k)!} \delta_{p+q,t-1} \delta_{p+k,2(s_1-1)} \delta_{p+k+2q,2(s_2-1)} c_{ij} \xi_{\alpha(p+q),\dot{\beta}(p+q)} \times \\
&\times [C^{i;\alpha(p)\gamma(k)\varphi\rho} H_{\varphi\rho} \varepsilon^{j;\alpha(q)}_{\gamma(k),\dot{\beta}(p+q)} - \bar{C}^{i;\dot{\beta}(p)\dot{\delta}(k)\dot{\varphi}\dot{\rho}} \bar{H}_{\dot{\varphi}\dot{\rho}} \varepsilon^{j;\alpha(p+q)}_{,\dot{\delta}(k)} \dot{\beta}(q)].
\end{aligned}$$

Thus,  $\delta J_{t,s_1,s_2}$  is exact on-shell. The same is true for the spin-one current (64).

As a consequence, the gauge transformation of  $Q_\xi$  is

$$\delta Q_\xi \simeq \int d \left( \sum_{m,n} \xi_{\alpha(m),\dot{\beta}(n)} H_{t,s_1,s_2}^{\alpha(m),\dot{\beta}(n)} \right) = 0.$$

Thus, though the current  $J_t$  is not gauge invariant, the corresponding charge is.



## 8 Conclusion

In this paper, HS currents  $J_{t,s_1,s_2}$  in  $AdS_4$ , built from boson fields of arbitrary spins obeying  $t \leq s_1 + s_2 - 1$  are found with the aid of the variation principle. Being represented as three-forms,  $J_{t,s_1,s_2}$  are closed but not exact hence leading to nontrivial HS charges. These charges are gauge invariant because  $\delta J_{t,s_1,s_2}$  is exact. In  $4d$  Minkowski case we have two series of currents: parity-even and “mysterious” parity-odd currents. In agreement with the conjecture of [10], we were not able to extend parity-odd currents to  $AdS_4$ .

Let us stress that the derivation of the currents via action applied in this paper leads to currents containing more than minimal number of derivatives with the higher-derivative terms corresponding to certain improvements. This is expected however since consistent cubic HS interactions are known to contain higher-derivative terms allowing to preserve HS gauge symmetries associated with gauge fields of different spins.

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